

# **The Quantum Vacuum**

*An Introduction to Quantum  
Electrodynamics*

**Peter W. Milonni**  
Los Alamos, New Mexico



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For Shwu-Fang, Moe, Nin, Carol Ann, and Betty

For no man can write anything who does not think that what he writes is for the time the history of the world; or do anything well who does not esteem his work to be of importance. My work may be of none, but I must not think of it as none, or I shall not do it with impunity.

In like manner, there is throughout nature something mocking, something that leads us on and on, but arrives nowhere; keeps no faith with us. All promise outruns the performance. We live in a system of approximations.

Ralph Waldo Emerson, *Essay on Nature* (1844).

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# Preface

According to present ideas there is no vacuum in the ordinary sense of tranquil nothingness. There is instead a fluctuating quantum vacuum. One purpose of this book is to survey some of our most important ideas about the quantum vacuum. A second is to describe, based on fundamental vacuum processes, the physical concepts of quantum electrodynamics (QED).

Why bother? Few people doubt the reality or significance of vacuum field fluctuations, and the formalism for perturbative QED calculations can already be found in many books. My answer is that, if QED is indeed the nonpareil physical theory, and if “the vacuum holds the key to a full understanding of the forces of nature,”<sup>1</sup> then it is worthwhile to look carefully at the *physical* ideas underlying QED vacuum effects, including not only such things as mass and charge renormalization, Lamb shifts and Casimir effects, but even more “elementary” things such as spontaneous emission, van der Waals forces, and the fundamental linewidth of a laser. Phenomena of the latter type, primarily nonrelativistic, are basic to quantum optics and other aspects of modern, applied QED. All of them involve the vacuum electromagnetic field in one way or another. All of them, furthermore, can be described physically in ways that involve *source* fields. A third purpose of this book is to exhibit and explain the relation between vacuum and source fields.

The modern view of the vacuum is closely related to zero-point energy, the energy associated with motion persisting even at the absolute zero of temperature, where classically all motion ceases. The idea of zero-point energy arose with the work of Planck and Einstein on the blackbody problem, and is connected to early, vague premonitions of wave-particle duality. Many physicists, including Pauli and Kramers, have been uncomfortable with zero-point energy, and even today the concept retains a peculiar flavor. We review the origins of the concept of zero-point energy in the first chapter.

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<sup>1</sup>P. C. W. Davies, *Superforce* (Simon and Schuster, New York, 1985), p. 104.

Chapters 2–8 are devoted to various aspects of the electromagnetic vacuum in nonrelativistic theory. In Chapter 2 we review the quantization of the electromagnetic field in the simplest way, by exploiting the mathematical equivalence of the field to a collection of harmonic oscillators. (A more sophisticated path is taken in Chapter 10.) The zero-point field energy associated with the source-free, vacuum field is a consequence of the zero-point energy of the simple harmonic oscillator. This zero-point energy is often ignored on the grounds that it is a constant addition to the field Hamiltonian and can therefore be eliminated by simply redefining the zero of energy. However, deletion of the zero-point field energy from the Hamiltonian does not eliminate it once and for all from consideration, for it “re-emerges” as the homogeneous solution of the operator Maxwell equations in the Heisenberg picture. In fact the vacuum field, with its zero-point energy, is *required* for the formal consistency of QED, and in particular to preserve canonical commutation relations.

In Chapter 2 we also introduce the Casimir force between two perfectly conducting plates in the standard way, by considering the change in the electromagnetic zero-point energy due to the presence of the plates. The Casimir effect was proposed at the same time (1948) that the Lamb shift and the anomalous magnetic moment of the electron were being interpreted in terms of zero-point field energy and fluctuations by Welton and others. I often wondered whether Casimir was at all influenced by these other developments, and in early 1992 I wrote to him with this question. Dr. Casimir has kindly given me permission to quote from his answer:<sup>2</sup>

No, I was not at all familiar with the work of Welton and others. I went my own, somewhat clumsy way . . .

A point I should have mentioned in later publications: Summer or autumn 1947 (but I am not absolutely certain that it was not somewhat earlier or later) I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that is something new. I told him that I was puzzled by the extremely simple form of the expressions for the interaction at very large distances and he mumbled something about zero-point energy. That was all, but it put me on a new track . . .

I found that calculating changes of zero-point energy really leads to the same results as the calculations of Polder and myself . . .

I do not think there were outside influences apart from those mentioned above. I did not myself contribute to further developments, nor to experimental confirmations (apart from proposing a crazy model for an electron, which did not lead to a value of the fine structure constant).

<sup>2</sup>H. B. G. Casimir, private communication, 12 March 1992.

In Chapter 3 we discuss various “vacuum fluctuation effects” — spontaneous emission, the (nonrelativistic) Lamb shift and anomalous moment, the Casimir and Casimir–Polder forces, and the van der Waals interaction — in elementary terms. These effects can be described and derived in different ways in terms of *source* fields. Different interpretations and derivations result from an *arbitrary* choice of ordering of photon annihilation and creation operators, as discussed in Chapter 4. In Chapter 5 we briefly review the nonrelativistic theory of radiation reaction, which underlies the source-field interpretation of “vacuum fluctuation effects.”

Chapter 6 shows how the vacuum field appears in various problems of quantum optics, cavity QED, and laser physics, including the fundamental quantum limit to the laser linewidth.

The Casimir effect plays such a prominent role in discussions of the quantum vacuum that we devote Chapters 7 and 8 to a rather detailed treatment of the forces between both conducting and dielectric bodies. We derive the Lifshitz expression for the force between two dielectric slabs from the perspectives of both vacuum and source fields, and show how various approaches are related. The macroscopic approach pioneered by Lifshitz is justified using the Ewald–Oseen extinction theorem. We discuss the experimental status of Casimir effects and some theoretical elaborations. We also describe Casimir’s “crazy model for an electron.”

Chapters 9 and 10 review some elementary features of the first-quantized Dirac equation and quantum field theory, respectively, including the relation between causality and spin statistics and the propagators for various quantum fields. We emphasize that *all quantum fields have zero-point energies and vacuum fluctuations*. In particular, the Dirac vacuum of electrons and positrons has a negative (infinite) zero-point energy, and this gives rise to Casimir effects analogous to those for the pure electromagnetic field.

Chapter 11 is an old-fashioned approach to relativistic self-energies and mass and charge renormalization. Some of the most important physical ideas of QED, including vacuum polarization, emerge rather clearly in such an approach. The modern approach, of course, is based principally on covariant perturbation theory and Feynman diagrams. In Chapter 12 we discuss the construction and utility of the diagrams, and again the calculation of basic things such as the electron self-energy, using both quantum field theory and the intuitive way of Feynman. The identification of some of the main physical ideas of relativistic QED in Chapter 11, *before* the introduction of Feynman diagrams, seems to me to have the advantage of avoiding confusion between the diagrams and the *physics*, although the distinction between the two is admittedly not easy to make when perturbative calculations are taken to high orders. Chapters 9, 10, and 12 provide a

painless short course, I think, for those crossing the relativistic Rubicon for the first or second time.

Originally I hoped to cover more ground, including chapters on the vacuum in quantum chromodynamics and cosmology. However, I gradually realized that in a book of the intended size I could not do this without sacrificing the rather considerable detail I wanted for various QED effects, particularly van der Waals and Casimir interactions. Since these things are not treated in any sort of detail in other books I know of, I decided to omit the somewhat more speculative aspects of vacuum theory and to focus on quantum electrodynamics.

Writing this book has given me much pleasure. I wish to thank especially Mei-Li Shih and Gordon W. F. Drake. Mei-Li not only was relentless in her encouragement but also checked many of the calculations and made good suggestions. In my opinion, but not hers, she is an author of this book. Professor Drake served as a reviewer for Academic Press and, by suggesting changes and identifying errors and non sequiturs, helped greatly to improve the quality of the manuscript.

Discussions relating to this book with Richard J. Cook, Joseph H. Eberly, Walter T. Grandy, Jr., Michael Lieber, David Nesbitt, M.D., Edwin A. Power, David H. Sharp, and Larry Spruch were enlightening and often encouraging. Professor Eberly supervised the Ph.D thesis work from which this book gradually evolved.

I think it is also appropriate to thank Gabriel Barton, Timothy H. Boyer, Trevor W. Marshall, and again, Edwin Power; their work has been partly responsible for my continued fascination with the subject.

Needless to say, H. B. G. Casimir has given us a lot to think about, and I thank him for his kind replies to my letters and for permission to quote from them.

Finally I wish to acknowledge the inestimable help I have received from the editors at Academic Press. I hope this book lives up to their high standards and expectations.

Peter W. Milonni  
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## Chapter 1

# Zero-Point Energy in Early Quantum Theory

The existence of a zero-point energy of size  $\frac{1}{2}h\nu$  [is] probable.  
— Albert Einstein and Otto Stern (1913)

### 1.1 Introduction

The importance of the blackbody problem in the development of quantum theory is recognized by every serious student of modern physics. What is not so widely known is that blackbody theory led also to the concept of zero-point energy, which was later to appear naturally in the mathematics of quantum theory. The relation of this energy to early premonitions of wave-particle dualism is similarly not widely appreciated. This chapter is a discussion of these roots of the concept of zero-point energy. We do not proffer any sort of rigorous historical analysis, but only a glimpse into some of the early physics of energy at the absolute zero of temperature.

### 1.2 The Blackbody Problem

In 1860 Kirchhoff derived a general relation between the radiative and absorptive strengths of a body held at a fixed temperature  $T$ . According to Kirchhoff's law the ratio of the radiative strength to the absorption coefficient for radiation of wavelength  $\lambda$  is the same for all bodies at temperature  $T$ , and defines a universal function  $F(\lambda, T)$ . This led to the abstraction of an ideal *blackbody* for which the absorption coefficient is unity at every



wavelength, corresponding to total absorption. Thus  $F(\lambda, T)$  characterizes the radiative strength at wavelength  $\lambda$  of a blackbody at temperature  $T$ . The problem was to determine the universal function  $F(\lambda, T)$ .

An important step was taken in 1884 by Boltzmann, who invoked several aspects of Maxwell's electromagnetic theory. The most important of these for the present discussion is the result that isotropic radiation exerts on a perfectly reflecting surface a pressure  $u/3$ , where  $u$  is the energy density of the radiation.<sup>1</sup> Boltzmann considered blackbody radiation confined in a cylinder of volume  $V$ , one end of which is a perfectly reflecting piston. The radiation pressure on the piston increases the volume by  $dV$ , and in order to maintain a constant temperature an amount of heat

$$dQ = dU + PdV = d(uV) + \frac{1}{3}udV = Vdu + \frac{4}{3}udV \quad (1.1)$$

must be added, according to the first law of thermodynamics. Kirchhoff's law implies that the total energy density  $u$  over all wavelengths is a function only of  $T$ , so that

$$dQ = V \frac{du}{dT} dT + \frac{4}{3}udV. \quad (1.2)$$

Associated with the expansion of the cylinder is an increase in entropy by

$$dS = \frac{1}{T}dQ = \frac{V}{T} \frac{du}{dT} dT + \frac{4}{3} \frac{u}{T} dV, \quad (1.3)$$

which, according to the second law of thermodynamics, is an exact differential. Thus

$$\frac{\partial S}{\partial T} = \frac{V}{T} \frac{du}{dT}, \quad \frac{\partial S}{\partial V} = \frac{4}{3} \frac{u}{T} \quad (1.4)$$

and

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} = \frac{1}{T} \frac{du}{dT} = \frac{4}{3} \frac{d}{dT} \left( \frac{u}{T} \right), \quad (1.5)$$

from which it follows that  $du/dT = 4u/T$  and

$$u = bT^4 \quad (\text{Stefan-Boltzmann law}), \quad (1.6)$$

where  $b$  is a universal constant. Stefan in 1879 had in fact suggested such a relation from an analysis of experimental data.

<sup>1</sup>A plane wave exerts a pressure  $2u$  on a reflecting surface on which it is normally incident. (See, for instance, W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, Mass., 1962), p. 193.) For plane waves propagating with equal intensities in both directions normal to the surface, this is reduced to  $u$ , or  $u/3$  if the radiation is isotropic.

The Stefan-Boltzmann law stands in conflict with elementary classical models of equilibrium between radiation and matter. Consider the classical oscillator model of an atom, where an electron is assumed to be bound by an elastic restoring force. If  $\rho(\nu)d\nu$  denotes the energy per unit volume of radiation in the frequency interval  $[\nu, \nu + d\nu]$ , then the rate at which the atom absorbs energy from the radiation field may be shown to be given by the formula (see Appendix A)

$$\dot{W}_A = \frac{\pi e^2}{3m} \rho(\nu_o), \quad (1.7)$$

where  $W_A$  is the electron energy,  $e$  and  $m$  are its charge and mass, respectively, and  $\nu_o$  is the natural oscillation frequency of the electron in the atom. The rate at which the electron radiates electromagnetic energy  $\dot{W}_{EM}$  is given by the well-known classical Larmor formula:

$$\dot{W}_{EM} = \frac{2e^2 a^2}{3c^3}, \quad (1.8)$$

where  $a$  is the acceleration of the electron. For oscillation at frequency  $\nu_o = \omega_o/2\pi$ ,  $a = -\omega_o^2 x$  and

$$\dot{W}_{EM} = \left( \frac{32\pi^4 e^2 \nu_o^4}{3c^3} \right) x^2, \quad (1.9)$$

where  $x$  is the electron displacement from its equilibrium position in the classical oscillator model of the atom. Now according to the virial theorem of classical mechanics the average potential energy  $\frac{1}{2}m\omega_o^2 x^2$  of the (one-dimensional) electron oscillator is equal to the average kinetic energy, and their sum is the total oscillator energy  $U$ . In a state of equilibrium between radiation and matter, furthermore, the energy absorption rate (1.7) should equal the emission rate (1.9). Thus

$$\rho(\nu_o) = \frac{8\pi\nu_o^2}{c^3} (m\omega_o^2 x^2) = \frac{8\pi\nu_o^2}{c^3} U, \quad (1.10)$$

or more generally

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3} U \quad (1.11)$$

for a blackbody, which absorbs at all frequencies  $\nu$ . Finally the equipartition theorem of classical statistical mechanics demands that the average value of  $U$  in thermal equilibrium is  $kT$ , where  $k$  is Boltzmann's constant, so that the spectral energy density of thermal radiation must be

$$\rho(\nu) = \left( \frac{8\pi\nu^2}{c^3} \right) kT \quad (\text{Rayleigh-Jeans distribution}). \quad (1.12)$$

The total electromagnetic energy density

$$u = \int_0^{\infty} \rho(\nu) d\nu \quad (1.13)$$

violates the Stefan-Boltzmann law. Furthermore the Rayleigh-Jeans law suffers from the ultraviolet catastrophe:  $u$  diverges when (1.12) is used for  $\rho(\nu)$ .

Equation (1.11) was derived by Planck and, as we shall see, played a very important role in his work on the blackbody problem.

Equation (1.12) for the spectral energy density of blackbody radiation was first deduced in a less explicit form by Rayleigh in 1900.<sup>2</sup> Although the derivation just outlined might be criticized for its reliance on a particular model of an atomic electron, it is easy to derive the Rayleigh-Jeans distribution on more general classical grounds. An electromagnetic field mode of frequency  $\nu$  is basically just a linear harmonic oscillator (see Chapter 2) that, according to the classical equipartition theorem, has an average energy  $kT$  at thermal equilibrium. Since the number of modes per unit volume in the frequency interval  $[\nu, \nu + d\nu]$  is  $(8\pi\nu^2/c^3)d\nu$ , the electromagnetic energy per unit volume in this frequency interval should be  $(8\pi\nu^2/c^3)(kT)d\nu = \rho(\nu)d\nu$ , which is the Rayleigh-Jeans law, independent of any particular model for the atoms with which the radiation is in thermal equilibrium. From this perspective the failure of classical theory, according to Kelvin and Rayleigh, must lie in its equipartition theorem.

Another classical result, due to Wien in 1893, must be mentioned. Wien basically followed Boltzmann's model of radiation contained in a cylinder with a piston, but included the Doppler shift of radiation reflected by the moving piston. This allowed radiant energy to be exchanged among different frequencies. Wien showed that the spectral energy density must follow the general form

$$\rho(\nu) = \nu^3 \phi_1(\nu/T) \quad (\text{Wien displacement law}), \quad (1.14)$$

or, in terms of wavelength,

$$\rho(\lambda) = \rho(\nu) \left| \frac{d\nu}{d\lambda} \right| = \lambda^{-5} \phi_2(\lambda T) \quad (\text{Wien displacement law}), \quad (1.15)$$

where  $\phi_1$  and  $\phi_2$  are undetermined functions. The Rayleigh-Jeans distribution obviously obeys Wien's "displacement law" (1.14).

<sup>2</sup>Motivated by Wien's work, Rayleigh also allowed for the possibility that a factor  $e^{-(\text{const})\nu/T}$  should be included, thus avoiding the ultraviolet catastrophe.

A few years later Wien presented arguments in support of the distribution

$$\rho(\lambda) = \alpha \lambda^{-5} e^{-\beta/\lambda T} \quad (\text{Wien distribution}), \quad (1.16)$$

where  $\alpha$  and  $\beta$  are constants. A similar distribution function, with the factor  $\lambda^{-5}$  replaced by  $\lambda^{-\gamma}$ , had just been proposed by Paschen as a fit to his experimental data. Paschen's data indicated that  $\gamma$  was between 5 and 6, thus providing some support for the displacement law. Further measurements showed that  $\gamma$  was indeed close to 5.

Wien's arguments for (1.16) seem to have been guided more by the desired result than by physics. To wit, he made the peculiar assumption that the wavelength and intensity of the radiation from a given atom (or molecule) are determined only by that atom's velocity. This allowed him to adduce the exponential term in (1.16) from the factor  $\exp(-mv^2/2kT)$  in the Maxwell-Boltzmann velocity distribution function. In any case the Wien distribution was soon to find a more secure provenance in Planck's work.

### 1.3 Planck's First Theory

Given that Planck was an expert in thermodynamics, it is not surprising that his work on the blackbody problem emphasized the concept of entropy. In a series of papers in the late 1890s, Planck produced a derivation of the Wien distribution from general thermodynamical considerations plus the *assumption* that the entropy of a collection of radiators depends only on their total energy. An important result was the following relation between the entropy  $S$  and average energy  $U$  of an elementary radiator (or "molecule" for our purposes) in thermal equilibrium with radiation at temperature  $T$ :

$$\frac{\partial^2 S}{\partial U^2} = -\frac{A}{U}, \quad (1.17)$$

where for a given radiator  $A$  is a constant. From this equation and the general relation  $\partial S/\partial U = 1/T$  it follows that

$$U = B e^{-1/AT}, \quad (1.18)$$

where  $B$  is another constant that, like  $A$ , may depend on the frequency of a given radiator. This result, together with (1.11), yields the radiation spectral energy density

$$\rho(\nu) = f(\nu) e^{-1/AT}, \quad (1.19)$$

where  $f(\nu)$  is some function of  $\nu$ . Wien's displacement law implies that  $f(\nu)$  and  $A$  are proportional to  $\nu^3$  and  $\nu^{-1}$ , respectively, so that

$$\rho(\nu) = C\nu^3 e^{-D\nu/T} \quad (C, D \text{ constants}) \quad (1.20)$$

or

$$\rho(\lambda) = \alpha\lambda^{-5} e^{-\beta/\lambda T} \quad (\alpha, \beta \text{ constants}), \quad (1.21)$$

which is the Wien distribution.

The Wien distribution, however, was soon found to be incorrect as experimentalists extended their spectral measurements to higher wavelengths. This was accomplished by the "residual rays" method, whereby longer wavelengths were isolated by multiple reflections off an appropriate crystal. In February 1900 Lummer and Pringsheim reported data that deviated from the Wien distribution by 40–50% for wavelengths between 12 and 18  $\mu\text{m}$ , and in October similar conclusions were reported by Rubens and Kurlbaum.

It was the work of his friend Rubens that led Planck to his formula for the spectral energy density of thermal radiation. In particular, the data indicated that  $\rho(\nu)$  was proportional to the temperature  $T$  for small  $\nu$  and large  $T$ . Planck found a formula with that behavior at small  $\nu$  and which approximated the Wien distribution for large  $\nu$ .

In a paper delivered at a meeting on 19 October Planck presented his formula and provided some justification for it.<sup>3</sup> For small  $\nu$  and large  $T$ , the experimental result  $\rho(\nu) \propto T$  and equation (1.11) imply  $U \propto T$  and therefore, since  $\partial S/\partial U = T^{-1}$ ,  $\partial^2 S/\partial U^2 \propto U^{-2}$  and  $S \propto \log U$ . On the other hand (1.17) leads to the Wien distribution, which has the correct form for large  $\nu$  and small  $T$ . Planck proposed the interpolation

$$\frac{\partial^2 S}{\partial U^2} = \frac{-A}{U(B+U)} \quad (A, B \text{ constants}). \quad (1.22)$$

According to Planck, equation (1.22) "is the simplest by far of all the expressions which yield  $S$  as a logarithmic function of  $U$  (a condition which probability theory suggests) and which besides coincides with the Wien law for small values of  $U$ ." Using again the relation  $\partial S/\partial U = 1/T$ , equation (1.11), and the Wien displacement law, one obtains from (1.22) the spectral energy density

$$\rho(\lambda) = \frac{\alpha\lambda^{-5}}{e^{\beta/\lambda T} - 1} \quad (\alpha, \beta \text{ constants}). \quad (1.23)$$

This formula was found to agree with all the existing data. In order to give it "a real physical meaning," Planck began what he later described as "a

<sup>3</sup>See the books by Kuhn and Pais and the articles by Klein cited at the end of the chapter.

few weeks of the most strenuous work of my life." The culmination of that work was the birth of quantum theory.

Planck's reasoning may be glibly summarized as follows. Consider  $N$  radiators of frequency  $\nu$  and total energy  $U_N = NU = P\epsilon$ , where  $P$  is a large integer and  $\epsilon$  is some finite element of energy. The entropy  $S_N = NS = k \log W_N$ , where  $W_N$  is the number of ways in which the  $P$  energy elements can be distributed among the  $N$  radiators. If  $N = P = 2$ , for instance, then the different partitions of the energy between the two radiators are  $(2\epsilon, 0)$ ,  $(\epsilon, \epsilon)$ , and  $(0, 2\epsilon)$  if the energy elements are assumed to be indistinguishable. Under this assumption we have, in general,

$$W_N = \frac{(N-1+P)!}{P!(N-1)!}, \quad (1.24)$$

which is the number of ways in which  $P$  indistinguishable balls can be put into  $N$  distinguishable boxes. Stirling's approximation ( $\log M! \cong M \log M - M$  for large  $M$ ) then gives, for  $N, P \gg 1$ ,

$$\begin{aligned} S &= \frac{k}{N} \log \frac{(N-1+P)!}{P!(N-1)!} \\ &\cong k \left[ \left(1 + \frac{P}{N}\right) \log \left(1 + \frac{P}{N}\right) - \frac{P}{N} \log \frac{P}{N} \right] \\ &= k \left[ \left(1 + \frac{U}{\epsilon}\right) \log \left(1 + \frac{U}{\epsilon}\right) - \frac{U}{\epsilon} \log \frac{U}{\epsilon} \right]. \end{aligned} \quad (1.25)$$

Thus

$$\frac{\partial S}{\partial U} = \frac{1}{T} = \frac{k}{\epsilon} \log \left(1 + \frac{\epsilon}{U}\right) \quad (1.26)$$

or

$$U = \frac{\epsilon}{e^{\epsilon/kT} - 1} \quad (1.27)$$

for the average energy of each radiator. The excellent agreement between (1.23) and experiment, together with equation (1.11), suggests that  $\epsilon$  is inversely proportional to the wavelength, or directly proportional to the frequency of the oscillator:

$$\epsilon = h\nu. \quad (1.28)$$

Then

$$U = \frac{h\nu}{e^{h\nu/kT} - 1} \quad (1.29)$$

and (1.11) implies

$$\rho(\nu) = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1} \quad (\text{Planck spectrum}) \quad (1.30)$$

for the spectral energy density of thermal radiation.

The expression (1.25) for  $S$  satisfies equation (1.22) with  $A = k$  and  $B = \epsilon$ . Once (1.25) is obtained, therefore, one is led to the form (1.23) for the spectral energy density. The great success of (1.23) in fitting the experimental data led Planck to what he later called an “act of desperation” needed to derive (1.25).

One aspect of this desperate act is the way Planck counted the number of ways, or “complexions,” in which  $P$  energy elements could be distributed among  $N$  radiators. His counting procedure was totally at odds with classical statistical methods in its treatment of the energy elements as fundamentally *indistinguishable*. In one sense Planck was following Boltzmann in regarding all complexions as equally likely, but of course his way of counting the number of complexions was radically different. His “energy elements” obeyed what would much later be recognized as Bose–Einstein statistics.

Another revolutionary (nonclassical) aspect of Planck’s calculation, of course, is the physical significance it attaches to the “energy elements” of size  $\epsilon$ , and the relation (1.28) between  $\epsilon$  and the frequency  $\nu$  of a *material* oscillator. Boltzmann had also employed “energy elements” in his counting of complexions, but in his calculations  $\epsilon$  had no particular significance and in fact could ultimately be taken to be zero once a formula for  $W_N$  had been obtained. If Planck had taken the limit  $\epsilon \rightarrow 0$  in equation (1.26), however, then  $\partial S/\partial U \rightarrow k/U$  and  $\partial^2 S/\partial U^2 \rightarrow -k/U^2$ , which leads to the Rayleigh–Jeans distribution. In Planck’s derivation of his spectrum, therefore, the quantization of energy was absolutely essential.

This is the traditional view of Planck’s innovation. It should be noted, however, that Kuhn (1978) has concluded that Planck did not in 1900 introduce any physical quantization of either radiation or material radiators. He argues that Planck’s radiators were simply “a device for bringing radiation to equilibrium, and it was justified, not by knowledge of the physical processes involved, but by Kirchhoff’s law, which made the equilibrium field independent of the equilibrium-producing material.”

Until about 1905 Planck’s formula was regarded as little more than a superb fit to the experimental data. Its true significance began to be appreciated only when it was realized that the Rayleigh–Jeans law was an inevitable consequence of classical physics and the equipartition theorem, and therefore that the blackbody experiments had uncovered a fundamental failure of known (classical) theory.

A curious circumstance relating to zero-point energy, which was noted by Einstein and Stern (1913), is worth mentioning. Consider the classical limit  $kT \gg h\nu$  of the expression (1.29) for the average energy of an

oscillator in thermal equilibrium with radiation:

$$U = \frac{h\nu}{e^{h\nu/kT} - 1} \cong \frac{h\nu}{1 + \frac{h\nu}{kT} + \frac{1}{2}\left(\frac{h\nu}{kT}\right)^2 - 1} = \frac{kT}{1 + \frac{1}{2}\frac{h\nu}{kT}} \cong kT - \frac{1}{2}h\nu. \quad (1.31)$$

Thus  $U$  contains a first-order temperature-independent correction to  $kT$ , the energy predicted by the equipartition theorem in the classical limit.

But

$$U + \frac{1}{2}h\nu = \frac{h\nu}{e^{h\nu/kT} - 1} + \frac{1}{2}h\nu, \quad (1.32)$$

which includes the zero-point energy  $\frac{1}{2}h\nu$ , does not have a first-order correction to  $kT$  in the classical limit. In Planck’s “second theory”  $U$  was in fact replaced by  $U + \frac{1}{2}h\nu$ .

## 1.4 Planck’s Zero-Point Energy

It was mentioned earlier that it took several years for the profound significance of Planck’s distribution to be appreciated. Planck himself was unsatisfied with the largely ad hoc theory he had used to derive his spectrum, and for many years he explored alternative hypotheses that might lead to it.

In 1912 Planck published his “second theory.” The absorption of radiation was assumed to proceed according to classical theory, whereas emission of radiation occurred discontinuously in discrete quanta of energy. Assume that an oscillator can radiate only after it has (continuously) absorbed an energy  $h\nu$ . Let  $P_n$  be the probability that it has energy between  $(n-1)h\nu$  and  $nh\nu$ . When, as a result of absorption of radiation, its energy reaches  $nh\nu$ , there is a probability  $p$  that it will lose *all* its energy in the form of radiation, and a probability  $1-p$  that it continues to absorb without emission of radiation. Thus  $P_2 = P_1(1-p)$ ,  $P_3 = P_2(1-p) = P_1(1-p)^2$ , ...,  $P_n = P_1(1-p)^{n-1}$ , and

$$\sum_{n=1}^{\infty} P_n = 1 = \sum_{n=1}^{\infty} P_1(1-p)^{n-1} = P_1/p \quad (1.33)$$

or  $P_1 = p$  is the probability that an oscillator in equilibrium with radiation has energy between 0 and  $h\nu$ ,  $P_2 = p(1-p)$  is the probability that it has energy between  $h\nu$  and  $2h\nu$ , and  $P_n = p(1-p)^{n-1}$  is the probability that it has energy between  $(n-1)h\nu$  and  $nh\nu$ . Following Boltzmann, Planck defines the oscillator entropy as

$$S = -k \sum_{n=1}^{\infty} P_n \log P_n = -k \sum_{n=1}^{\infty} p(1-p)^{n-1} \log[p(1-p)^{n-1}]$$

$$= -k \left[ \frac{1}{p} \log p + \left( \frac{1}{p} - 1 \right) \log \left( \frac{1}{p} - 1 \right) \right]. \quad (1.34)$$

Planck now assumes that all energies between  $(n-1)h\nu$  and  $nh\nu$  are equally likely, so that the average energy of the oscillators with energy between  $(n-1)h\nu$  and  $nh\nu$  is  $\frac{1}{2}(n+n-1)h\nu = (n-\frac{1}{2})h\nu$ . The average oscillator energy is then

$$U = \sum_{n=1}^{\infty} (n - \frac{1}{2})h\nu P_n = h\nu \sum_{n=1}^{\infty} (n - \frac{1}{2})p(1-p)^{n-1} = \left( \frac{1}{p} - \frac{1}{2} \right)h\nu \quad (1.35)$$

or  $1/p = U/h\nu + \frac{1}{2}$ . From (1.34), therefore,

$$S = k \left[ \left( \frac{U}{h\nu} + \frac{1}{2} \right) \log \left( \frac{U}{h\nu} + \frac{1}{2} \right) - \left( \frac{U}{h\nu} - \frac{1}{2} \right) \log \left( \frac{U}{h\nu} - \frac{1}{2} \right) \right]. \quad (1.36)$$

Using once again the relation  $\partial S/\partial U = 1/T$ , Planck obtained

$$U = \frac{1}{2}h\nu \frac{e^{h\nu/kT} + 1}{e^{h\nu/kT} - 1} = \frac{h\nu}{e^{h\nu/kT} - 1} + \frac{1}{2}h\nu. \quad (1.37)$$

This implies that  $U \neq 0$  when  $T \rightarrow 0$ : when  $T \rightarrow 0$ ,  $U \rightarrow \frac{1}{2}h\nu$ . Planck's equation (1.37) marked the birth of the concept of zero-point energy.

To derive  $\rho(\nu)$  Planck could not resort to equation (1.11), since the derivation of that equation assumed continuous absorption and emission processes. Instead he made the assumption that the ratio of the probability that an oscillator does not emit radiation, to the probability that it does, is proportional to  $\rho(\nu)$ :  $(1-p)/p = C\rho(\nu)$ , or  $1/p = C\rho(\nu) + 1$ , where  $C$  is a constant of proportionality. This assumption is plausible in that, the greater the radiation intensity, the more absorption should dominate emission. (Planck, of course, was not at this time aware of the possibility of stimulated emission!) Then, from (1.35),  $U = [C\rho(\nu) + \frac{1}{2}]h\nu$  or

$$\rho(\nu) = \frac{1}{C} \frac{1}{e^{h\nu/kT} - 1}. \quad (1.38)$$

To determine  $C$  Planck appeals to the classical limit, where the Rayleigh-Jeans law should apply: for  $kT \gg h\nu$ ,  $\rho(\nu)$  should reduce to (1.12), which requires that  $1/C = 8\pi h\nu^3/c^3$  and therefore that

$$\rho(\nu) = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1}. \quad (1.39)$$

It is interesting that, in deducing  $C$  in this way, Planck was employing what would soon come to be called the *correspondence principle*. Furthermore

Planck's probability  $p$  might well be regarded as the first example of a quantum transition probability.

It is also noteworthy that in Planck's second theory the material oscillators have zero-point energy but the electromagnetic field does not:  $\rho(\nu) \rightarrow 0$  for  $T \rightarrow 0$ . Had Planck simply used equation (1.11) to relate  $\rho(\nu)$  and  $U$ , he would have obtained from (1.37) the spectral energy density

$$\rho'(\nu) = \rho(\nu) + 4\pi h\nu^3/c^3 = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1} + 4\pi h\nu^3/c^3, \quad (1.40)$$

which, as we will see later, turns out to be the correct spectrum from the standpoint of modern quantum electrodynamics. The zero-point energy appearing in Planck's expression (1.37) is also perfectly correct according to modern theory, even though Planck's route to it is not.

By 1914 Planck was convinced that zero-point energy would be of no experimental consequence. However, the concept attracted much attention, and soon came to play a major role in the work of Einstein.

## 1.5 The Einstein-Hopf Model

"Concerning a Heuristic Point of View Toward the Emission and Transformation of Light," Einstein (1905) deduced that radiation satisfying the Wien distribution "behaves thermodynamically as though it consisted of a number of independent energy quanta of magnitude  $[h\nu]$ ." Based on this viewpoint he *predicted* the linear relation between radiation frequency and stopping potential in the photoelectric effect, a prediction confirmed by Millikan's experiments in 1916. In 1906 he argued that "in emission and absorption the energy of a [Planck oscillator] changes by jumps which are integral multiples of  $h\nu$ ." These were the beginnings of the photon concept.

Einstein struggled with the blackbody problem for more than ten years after he introduced his heuristic viewpoint concerning energy quanta of radiation. In one important paper Einstein and Hopf (1910b) studied a simple model for the thermal equilibrium between oscillating dipoles and electromagnetic radiation. Imagine each dipole to consist of a particle of mass  $m$  and charge  $e$ , bound by an elastic restoring force to a mass  $M (>> m)$  of opposite charge. The equation of motion for a linear dipole oscillator is then (see Appendix A)

$$\frac{d^2z}{dt^2} + \omega_o^2 z - \tau \ddot{z} = \frac{e}{m} E_z(t), \quad (1.41)$$

where  $\omega_o (= 2\pi\nu_o)$  is the natural oscillation frequency,  $E_z(t)$  is the  $z$ -component of the external electric field acting on the particle,  $\tau \ddot{z}$  is the

radiation reaction term, and  $\tau = 2e^2/3mc^3$ . The two oppositely charged particles define an electric dipole moment  $ez(t)$ . Implicit in equation (1.41) is the *electric dipole approximation* of neglecting any spatial variation of  $E_z(t)$  over the distance separating the particles. It is also assumed that the interaction of the dipole with the magnetic field is negligible.

Equation (1.41) is essentially the same equation used earlier by Planck to derive equation (1.11) (Appendix A). In the Einstein–Hopf model, however, the dipole oscillators of mass  $M + m \cong M$  are allowed to move; for simplicity they are constrained to move only along the  $x$  axis. Einstein and Hopf showed that there is a retarding force on a moving dipole as a result of its interaction with the field. This force acts to decrease its kinetic energy. Due to recoil associated with emission and absorption, however, the field also acts to increase the kinetic energy of a dipole. The condition for equilibrium is that the increase in kinetic energy due to recoil balances the decrease in kinetic energy associated with the retarding force.

Assuming  $v/c \ll 1$ , Einstein and Hopf showed that the retarding force due to motion through a thermal field of spectral energy density  $\rho(\omega_o)$  is

$$F = -Rv, \quad (1.42)$$

where

$$R = \frac{4\pi^2 e^2}{5mc^2} \left[ \rho(\omega_o) - \frac{\omega_o}{3} \frac{d\rho}{d\omega_o} \right] \quad (1.43)$$

and  $v$  is the velocity of the dipole. Essentially this same result is derived in Appendix B.

Consider now a dipole with linear momentum  $Mv(t)$  at time  $t$ . After a short time  $\delta t$  its momentum is

$$Mv(t + \delta t) = Mv(t) + \Delta - Rv(t)\delta t, \quad (1.44)$$

where  $\Delta$  is the impulse imparted to the dipole in the time interval  $\delta t$  as a result of recoil associated with emission and absorption of radiation. Then

$$M^2 v^2(t + \delta t) - M^2 v^2(t) = \Delta^2 - 2MRv^2(t)\delta t + (2M - R\delta t)v(t)\Delta \quad (1.45)$$

when  $\delta t$  is taken to be small enough (or  $M$  large enough) that terms quadratic in  $\delta t$  are negligible. Now take the equilibrium ensemble average of both sides of (1.45):

$$2M \left[ \left\langle \frac{1}{2} Mv^2(t + \delta t) \right\rangle - \left\langle \frac{1}{2} Mv^2(t) \right\rangle \right] = 0 = \langle \Delta^2 \rangle - 4R\delta t \left\langle \frac{1}{2} Mv^2(t) \right\rangle. \quad (1.46)$$

In writing this expression we have used the fact that  $\langle v(t)\Delta \rangle = 0$ , since  $\Delta$  is equally likely to be positive or negative in the time interval from  $t$

to  $t + \delta t$ . In thermal equilibrium, furthermore, the equipartition theorem requires the average kinetic energy to be  $\langle \frac{1}{2} Mv^2(t) \rangle = \frac{1}{2} kT$ . The condition for thermal equilibrium is therefore

$$(\delta t)^{-1} \langle \Delta^2 \rangle = 2RkT. \quad (1.47)$$

It remains to determine  $\langle \Delta^2 \rangle$ .

The force on an electric dipole moment  $e\hat{z}z(t)$  in the Einstein–Hopf model, where the dipole points in the  $z$  direction and is free to move only along  $x$ , may for our purposes be taken to be  $F_x = e\hat{z}z(t)\partial E_z(t)/\partial x$ .<sup>4</sup> The impulse imparted to the particle during the time interval from  $t = 0$  to  $t = \delta t$  is thus

$$\Delta = e \int_0^{\delta t} dt z(t) \frac{\partial E_z(t)}{\partial x}. \quad (1.48)$$

Einstein and Hopf write the electric field as a superposition of plane waves with independent random phases  $\theta_{\mathbf{k}\lambda}$ :

$$\mathbf{E}(\mathbf{r}, t) = i \sum_{\mathbf{k}\lambda} [A_{\mathbf{k}\lambda} e^{-i(\omega_{\mathbf{k}} t + \theta_{\mathbf{k}\lambda})} - A_{\mathbf{k}\lambda}^* e^{i(\omega_{\mathbf{k}} t + \theta_{\mathbf{k}\lambda})}] \mathbf{e}_{\mathbf{k}\lambda}, \quad (1.49)$$

where  $\mathbf{e}_{\mathbf{k}\lambda}$  is a unit polarization vector for a plane wave with wave vector  $\mathbf{k}$  and linear polarization index  $\lambda$  ( $= 1, 2$ ). The steady-state solution of equation (1.41) is then

$$z(t) = \frac{ie}{m} \sum_{\mathbf{k}\lambda} [F_{\mathbf{k}\lambda} e^{-i(\omega_{\mathbf{k}} t + \theta_{\mathbf{k}\lambda})} - F_{\mathbf{k}\lambda}^* e^{i(\omega_{\mathbf{k}} t + \theta_{\mathbf{k}\lambda})}], \quad (1.50)$$

where the origin of coordinates has been chosen to be at the position of the dipole and  $F_{\mathbf{k}\lambda} = -A_{\mathbf{k}\lambda} \mathbf{e}_{\mathbf{k}\lambda z} [\omega_{\mathbf{k}}^2 - \omega_o^2 + i\tau\omega_{\mathbf{k}}^3]^{-1}$ , where  $\mathbf{e}_{\mathbf{k}\lambda z}$  is the  $z$ -component of  $\mathbf{e}_{\mathbf{k}\lambda}$ . In a separate paper Einstein and Hopf (1910a) show that  $E_z(t)$  and  $\partial E_z/\partial x$  must be treated as independent random variables in the time integral (1.48). It then follows from (1.48)–(1.50) by straightforward manipulations that  $\langle \Delta \rangle = 0$  and

$$(\delta t)^{-1} \langle \Delta^2 \rangle = \left( \frac{4\pi^4 c^4 \tau}{5\omega_o^2} \right) \rho^2(\omega_o), \quad (1.51)$$

where the ensemble average is taken over the random phases  $\theta_{\mathbf{k}\lambda}$ .

Equation (1.47), together with (1.43) and (1.51), now gives a differential equation that must be satisfied by the spectral energy density of thermal radiation:

$$\rho(\omega) - \frac{\omega}{3} \frac{d\rho}{d\omega} = \left( \frac{\pi^2 c^3}{3\omega^2 kT} \right) \rho^2(\omega). \quad (1.52)$$

<sup>4</sup> See P. W. Milonni and M.-L. Shih, *Am. J. Phys.* 59, 684 (1991).

The solution of this equation satisfying  $\rho(0) = 0$  is

$$\rho(\omega) = \frac{\omega^2 kT}{\pi^2 c^3}, \quad (1.53)$$

which is seen to be just the Rayleigh–Jeans law when we recall that  $\omega = 2\pi\nu$  and  $\rho(\omega) = \rho(\nu)/2\pi$ .

The beautifully cogent arguments of Einstein and Hopf provide further evidence that the Rayleigh–Jeans law is an inexorable consequence of classical physics. However, we shall see that their results are dramatically altered when zero-point energy is postulated.

## 1.6 Einstein and Stern's Zero-Point Energy

In 1913 Einstein and Stern noted that an ad hoc postulate about zero-point energy in the Einstein–Hopf model would lead to the Planck spectrum. First let us note that equation (1.11) allows us to write (1.52) in a form in which the average dipole energy  $U$  appears explicitly:

$$\rho(\omega) - \frac{\omega}{3} \frac{d\rho}{d\omega} = \frac{1}{3kT} \rho(\omega) U. \quad (1.54)$$

Now suppose the average oscillator energy  $U$  is replaced by  $U + \hbar\omega$ . This means that the dipole oscillators are now assumed to have a zero-point energy  $\hbar\omega$ . Equation (1.54) is then replaced by

$$\begin{aligned} \rho(\omega) - \frac{\omega}{3} \frac{d\rho}{d\omega} &= \frac{1}{3kT} \rho(\omega) U + \frac{\hbar\omega}{3kT} \rho(\omega) \\ &= \frac{\pi^2 c^3}{3\omega^2 kT} \rho^2(\omega) + \frac{\hbar\omega}{3kT} \rho(\omega) \\ &= \frac{\pi^2 c^3}{3\omega^2 kT} [\rho^2(\omega) + \frac{\hbar\omega^3}{\pi^2 c^3} \rho(\omega)]. \end{aligned} \quad (1.55)$$

The solution of this equation satisfying  $\rho(0) = 0$  is

$$\rho(\omega) = \frac{\hbar\omega^3/\pi^2 c^3}{e^{\hbar\omega/kT} - 1} \quad (\text{Planck spectrum}). \quad (1.56)$$

In other words, if it is assumed that the dipole oscillators in the Einstein–Hopf model have a zero-point energy  $\hbar\omega$ , then the equilibrium spectrum of radiation is found to be the Planck spectrum.

The oscillator zero-point energy postulated by Einstein and Stern is twice that found earlier by Planck. Since we now know that Planck's zero-point energy  $\frac{1}{2}\hbar\omega$  is the correct one, it is interesting to see how Einstein and Stern arrived at the correct spectrum using the wrong zero-point energy.

According to quantum theory a field mode of frequency  $\omega$ , like a material oscillator, has a zero-point energy  $\frac{1}{2}\hbar\omega$  (see Chapter 2). The total zero-point energy of a linear dipole oscillator of frequency  $\omega$  and a field mode of the same frequency is therefore  $\frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega = \hbar\omega$ . Einstein and Stern's zero-point energy  $\hbar\omega$  is just this, *but they attributed it solely to the material dipole oscillators.*

Suppose we include in the Einstein–Hopf model a zero-point energy  $\frac{1}{2}\hbar\omega$  for a dipole oscillator *and* a zero-point energy  $\frac{1}{2}\hbar\omega$  for each field mode. Since there are  $(8\pi\nu^2/c^3)d\nu = (\omega^2/\pi^2 c^3)d\omega$  field modes per unit volume in the frequency interval  $[\omega, \omega + d\omega]$ , the spectral energy density of the zero-point field is

$$\rho_o(\omega) = (\omega^2/\pi^2 c^3) \frac{1}{2} \hbar\omega = \frac{\hbar\omega^3}{2\pi^2 c^3}. \quad (1.57)$$

If we replace  $\rho(\omega)$  in (1.54) by  $\rho(\omega) + \rho_o(\omega)$ , the left side is unchanged:

$$[\rho(\omega) + \rho_o(\omega)] - \frac{\omega}{3} \frac{d}{d\omega} [\rho(\omega) + \rho_o(\omega)] = \rho(\omega) - \frac{\omega}{3} \frac{d\rho}{d\omega}. \quad (1.58)$$

If we also account for the zero-point energy of the dipole oscillators by replacing  $U$  by  $U + \frac{1}{2}\hbar\omega$ , the product  $\rho(\omega)U$  on the right side of (1.54) is changed to

$$\begin{aligned} [\rho(\omega) + \rho_o(\omega)][U + \frac{1}{2}\hbar\omega] &= \rho(\omega)U + \frac{1}{2}\hbar\omega\rho(\omega) \\ &\quad + \rho_o(\omega)U + \frac{1}{2}\hbar\omega\rho_o(\omega) \\ &= \frac{\pi^2 c^3}{\omega^2} [\rho^2(\omega) + \rho_o(\omega)\rho(\omega) + \frac{\hbar\omega^3}{2\pi^2 c^3} \rho(\omega)] + \frac{1}{2}\hbar\omega\rho_o(\omega) \\ &= \frac{\pi^2 c^3}{\omega^2} [\rho^2(\omega) + \frac{\hbar\omega^3}{\pi^2 c^3} \rho(\omega)] + \frac{1}{2}\hbar\omega\rho_o(\omega), \end{aligned} \quad (1.59)$$

where we have used (1.57) and (1.11) in the form  $U = (\pi^2 c^3/\omega^2)\rho(\omega)$ .

The term  $\frac{1}{2}\hbar\omega\rho_o(\omega)$  in (1.59) results from a coupling of the zero-point motion of a dipole oscillator to the zero-point oscillations of the field. In quantum theory, in effect, no such coupling arises: an oscillator in its ground state in the absence of any applied field remains in its ground state. We shall see later how this comes about, but for now let us just accept it and drop the term  $\frac{1}{2}\hbar\omega\rho_o(\omega)$  in (1.59):

$$[\rho(\omega) + \rho_o(\omega)][U + \frac{1}{2}\hbar\omega] \rightarrow \frac{\pi^2 c^3}{\omega^2} [\rho^2(\omega) + \frac{\hbar\omega^3}{\pi^2 c^3} \rho(\omega)]. \quad (1.60)$$

From (1.54), (1.58), and (1.60), then, we have

$$\rho(\omega) - \frac{\omega}{3} \frac{d\rho}{d\omega} = \frac{\pi^2 c^3}{3\omega^2 kT} [\rho^2(\omega) + \frac{\hbar\omega^3}{\pi^2 c^3} \rho(\omega)], \quad (1.61)$$

which is exactly the Einstein–Stern equation (1.55). The complete spectrum  $\rho(\omega) + \rho_o(\omega)$  is then given by equation (1.40).

This route to the Planck spectrum may be summarized as follows. We modified the Einstein–Hopf model to include a zero-point energy  $\frac{1}{2}\hbar\omega$  for a dipole oscillator and a zero-point energy  $\frac{1}{2}\hbar\omega$  for each mode of the electromagnetic field, and anticipated a result of quantum theory that there is no contribution from the coupling of the zero-point oscillations of the dipole and the field. This led to the Einstein–Stern equation (1.55). Einstein and Stern, however, did not invoke any zero-point energy of the field, and to arrive at the Planck spectrum their dipole oscillators had to have a zero-point energy  $\frac{1}{2}\hbar\omega$  plus what we now know to be the zero-point energy of a field mode of the same frequency.

Why did Einstein and Stern not assume zero-point energy for the field? After all, one might have thought that the relation (1.11) between  $\rho(\omega)$  and  $U$  would have made it obvious that, if either the dipole oscillator or the field has a zero-point energy, then so must the other. If Planck’s zero-point energy  $\frac{1}{2}\hbar\omega$  is added to  $U$  in (1.11), for instance, then for consistency we must add the spectral energy density  $\rho_o(\omega)$  of the zero-point field to  $\rho(\omega)$ :

$$\rho(\omega) + \rho_o(\omega) = \frac{\omega^2}{\pi^2 c^3} \left( U + \frac{1}{2} \hbar \omega \right), \quad (1.62)$$

or again  $\rho_o(\omega) = \hbar\omega^3/2\pi^2 c^3$ , which in turn implies that each field mode has a zero-point energy  $\frac{1}{2}\hbar\omega$ .

However, such a “consistency” argument rests on the usual acuity of hindsight. The fact is that at various stages in Einstein’s long efforts to understand the Planck spectrum he seriously doubted the general validity of Planck’s equation (1.11). This is not surprising, for if Planck had simply invoked equipartition of energy and used  $U = kT$  in (1.11), he would have obtained the Rayleigh–Jeans spectrum. It is not clear whether Planck was even aware at the time of the classical equipartition theorem. If he had known and believed the equipartition theorem, as Einstein later remarked, “he would probably not have made his great discovery.”<sup>5</sup>

There is another reason why Einstein and Stern might have been unwilling to attribute a zero-point energy to the field: if  $\rho(\omega)$  and  $U$  are replaced by  $\rho(\omega) + \rho_o(\omega)$  and  $U + \frac{1}{2}\hbar\omega$ , respectively, in the Einstein–Hopf model, then one obtains the Rayleigh–Jeans spectrum for the total spectral density  $\rho(\omega) + \rho_o(\omega)$ . Crucial to the derivation of the Planck spectrum is the omission of the term  $\frac{1}{2}\hbar\omega\rho_o(\omega)$  in (1.59). This omission occurs automatically in the quantum theory of the Einstein–Hopf model, as we shall see in the next

chapter. Without this consequence of quantum theory available to them, Einstein and Stern may have simply discounted the possibility of zero-point electromagnetic energy. Indeed, the first suggestion that there might be a zero-point electromagnetic field is due not to Planck or Einstein and Stern, but to Nernst (1916).

## 1.7 Einstein’s Fluctuation Formula

Prior to his work with Hopf and Stern, Einstein (1909) had derived a formula for the energy fluctuations of thermal radiation. Denoting the variance in energy in the volume  $V$  and in the frequency interval  $[\omega, \omega + d\omega]$  by  $\langle \Delta E_\omega^2 \rangle$ , we may write the Einstein fluctuation formula as

$$\langle \Delta E_\omega^2 \rangle = [\hbar\omega\rho(\omega) + \frac{\pi^2 c^3}{\omega^2} \rho^2(\omega)] V d\omega. \quad (1.63)$$

The importance of this formula lies in Einstein’s interpretation of it. The first term in brackets, according to Einstein, may be obtained “if radiation were to consist of independently moving pointlike quanta of energy  $h\nu$ ”:

$$\langle \Delta E_\omega^2 \rangle_{\text{particles}} = \hbar\omega\rho(\omega) V d\omega, \quad (1.64)$$

whereas the second term follows when the field is treated as a superposition of independently fluctuating waves:

$$\langle \Delta E_\omega^2 \rangle_{\text{waves}} = \frac{\pi^2 c^3}{\omega^2} \rho^2(\omega) V d\omega. \quad (1.65)$$

Thus  $\langle \Delta E_\omega^2 \rangle$  has both wave and particle contributions. The Einstein fluctuation formula was the earliest indicator of the wave-particle dualism in quantum theory.

The “wave” term (1.65) may be derived from the superposition (1.49) of waves with independent random phases. For instance,

$$\begin{aligned} \langle E^2(\mathbf{r}, t) \rangle &= -2\text{Re} \sum_{\mathbf{k}_1 \lambda_1} \sum_{\mathbf{k}_2 \lambda_2} [A_{\mathbf{k}_1 \lambda_1} A_{\mathbf{k}_2 \lambda_2} e^{-i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2})t} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} \\ &\quad \times \langle e^{-i(\theta_{\mathbf{k}_1 \lambda_1} + \theta_{\mathbf{k}_2 \lambda_2})} \rangle - A_{\mathbf{k}_1 \lambda_1} A_{\mathbf{k}_2 \lambda_2}^* e^{-i(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t} \\ &\quad \times e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}} \langle e^{-i(\theta_{\mathbf{k}_1 \lambda_1} - \theta_{\mathbf{k}_2 \lambda_2})} \rangle] e_{\mathbf{k}_1 \lambda_1} \cdot e_{\mathbf{k}_2 \lambda_2}, \end{aligned} \quad (1.66)$$

where again the average is over the phases  $\theta_{\mathbf{k}\lambda}$ , which are assumed to be independent, uniformly distributed random variables on the interval  $[0, 2\pi]$ .

<sup>5</sup> Albert Einstein: *Philosopher-Scientist*, ed. P. A. Schilpp (Tudor, New York, 1949), p. 43.